Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

2 - 11 Cauchy-Riemann equations
Are the following functions analytic? Use (1) on p. 625 or (7) on p. 628.

3. f[z] = e^{-2x} (Cos[2 y] - i Sin[2 y])

```
In[1]:= Clear["Global`*"]

In[2]:= f[x_, y_] = e^(-2 x) (Cos[2 y] - i Sin[2 y])
Out[2]= e^(-2 x) (Cos[2 y] - i Sin[2 y])
```

The test for analyticity for complex functions comes from the Cauchy-Riemann criterion.

```
In[3]:= u[x_, y_] = e^(-2 x) Cos[2 y]
Out[3]= e^(-2 x) Cos[2 y]

In[5]:= v[x_, y_] = -e^(-2 x) Sin[2 y]
Out[5]= -e^(-2 x) Sin[2 y]

In[6]:= PossibleZeroQ[D[u[x, y], x] - D[v[x, y], y]]
        PossibleZeroQ[D[v[x, y], x] + D[u[x, y], y]]
Out[6]= True

Out[7]= True
```

The function f passes the C-R test and is therefore analytic, yes.

5. f[z] = Re[z^2] - i Im[z^2]

```
In[8]:= Clear["Global`*"]

In[9]:= z = x + i y
Out[9]= x + i y

In[10]:= f[x_, y_] = Re[z^2] - i Im[z^2]
Out[10]= -i Im[(x + i y)^2] + Re[(x + i y)^2]

In[11]:= ComplexExpand[f[x, y]]
Out[11]= x^2 - 2 i x y - y^2

In[12]:= u[x_, y_] = x^2 - y^2
Out[12]= x^2 - y^2
```

In[13]:= v[x_, y_] = -2 x y

Out[13]= $-2\,x\,y$

In[14]:= PossibleZeroQ[D[u[x, y], x] - D[v[x, y], y]]
PossibleZeroQ[D[v[x, y], x] + D[u[x, y], y]]

Out[14]= False

Out[15]= False

PZQs are not both true in this case, therefore f is not analytic, no.

7. $f[z] = \frac{i}{z^8}$

In[16]:= Clear["Global`*"]

In[17]:= z = x + i y

Out[17]= $x + i\,y$

In[18]:= $f[x\_, y\_] = \frac{i}{z^8}$

Out[18]= $\frac{i}{(x + i\,y)^8}$

In[19]:= ComplexExpand[f[x, y]]

Out[19]= $$\frac{8 x^7 y}{\left(x^2+y^2\right)^8} - \frac{56 x^5 y^3}{\left(x^2+y^2\right)^8} + \frac{56 x^3 y^5}{\left(x^2+y^2\right)^8} - \frac{8 x y^7}{\left(x^2+y^2\right)^8} + i\left(\frac{x^8}{\left(x^2+y^2\right)^8} - \frac{28 x^6 y^2}{\left(x^2+y^2\right)^8} + \frac{70 x^4 y^4}{\left(x^2+y^2\right)^8} - \frac{28 x^2 y^6}{\left(x^2+y^2\right)^8} + \frac{y^8}{\left(x^2+y^2\right)^8}\right)$$

In[20]:= $$u[x\_, y\_] = \frac{8 x^7 y}{\left(x^2+y^2\right)^8} - \frac{56 x^5 y^3}{\left(x^2+y^2\right)^8} + \frac{56 x^3 y^5}{\left(x^2+y^2\right)^8} - \frac{8 x y^7}{\left(x^2+y^2\right)^8};$$

In[21]:= $$v[x\_, y\_] = \frac{x^8}{\left(x^2+y^2\right)^8} - \frac{28 x^6 y^2}{\left(x^2+y^2\right)^8} + \frac{70 x^4 y^4}{\left(x^2+y^2\right)^8} - \frac{28 x^2 y^6}{\left(x^2+y^2\right)^8} + \frac{y^8}{\left(x^2+y^2\right)^8};$$

In[22]:= PossibleZeroQ[D[u[x, y], x] - D[v[x, y], y]]
PossibleZeroQ[D[v[x, y], x] + D[u[x, y], y]]

Out[22]= True

Out[23]= True

From the problem description in can be seen that z cannot be 0; otherwise, since PZQs are both true, the expression f[z] is analytic, yes.

9. $f[z] = \frac{3\pi^2}{z^3 + 4\pi^2 z}$

In[24]:= Clear["Global`*"]

In[25]:= $f[z] = \frac{3\pi^2}{z^3 + 4\pi^2 z}$

Out[25]= $\frac{3\pi^2}{4\pi^2 z + z^3}$

In[26]:= ff[x_, y_] = f[z] /. z → x + ⅈ y

Out[26]= $\frac{3\pi^2}{4\pi^2 (x + i y) + (x + i y)^3}$

In[27]:= dr = ComplexExpand[Re[ff[x, y]]];

In[28]:= u[x_, y_] = FullSimplify[dr]

Out[28]= $\frac{3\pi^2 x \left(4\pi^2 + x^2 - 3y^2\right)}{\left(x^2 + y^2\right)\left(16\pi^4 + 8\pi^2 (x - y)(x + y) + \left(x^2 + y^2\right)^2\right)}$

In[29]:= di = ComplexExpand[Im[ff[x, y]]];

In[30]:= v[x_, y_] = FullSimplify[di]

Out[30]= $\frac{3\pi^2 y \left(-4\pi^2 - 3x^2 + y^2\right)}{\left(x^2 + y^2\right)\left(16\pi^4 + 8\pi^2 (x - y)(x + y) + \left(x^2 + y^2\right)^2\right)}$

```
In[31]:= PossibleZeroQ[D[u[x, y], x] - D[v[x, y], y]]
         PossibleZeroQ[D[v[x, y], x] + D[u[x, y], y]]
Out[31]= True
Out[32]= True
```

Here is another case where z is not allowed to equal zero; with that exception, PZQs are both true, so the function is judged analytic, yes.

11. f[z] = Cos[x] Cosh[y] - I Sin[x] Sinh[y]

```
In[33]:= Clear["Global`*"]

In[34]:= f[x_, y_] = Cos[x] Cosh[y] - I Sin[x] Sinh[y]
Out[34]= Cos[x] Cosh[y] - ⅈ Sin[x] Sinh[y]

In[35]:= f[z] = Cos[x] Cosh[y] - I Sin[x] Sinh[y]
Out[35]= Cos[x] Cosh[y] - ⅈ Sin[x] Sinh[y]
```

```
In[36]:= u[x_, y_] = Cos[x] Cosh[y]
Out[36]= Cos[x] Cosh[y]

In[37]:= v[x_, y_] = -Sin[x] Sinh[y]
Out[37]= -Sin[x] Sinh[y]

In[38]:= PossibleZeroQ[D[u[x, y], x] - D[v[x, y], y]]
         PossibleZeroQ[D[v[x, y], x] + D[u[x, y], y]]
Out[38]= True
Out[39]= True
```

In this case there are no domain restrictions, and the Cauchy-Riemann test is passed by u and v, yes.

Are the following functions harmonic? If your answer is yes, find a corresponding analytic function **f[z]=u[x,y]+iv[x,y].**

13. u = x y

```
Clear["Global`*"]

u[x_, y_] = x y
x y

Simplify[Laplacian[u[x, y], {x, y}]]
0
```

The function u passes the test for harmonic function. Now to look for a corresponding analytic function.

```
D[u[x, y], x]
y
```

$v_y = u_x = y$ and $v_x = -u_y = -x$

according to the Cauchy-Riemann criteria, which I must follow. Integrating the first equation with respect to y and differentiating the result with respect to x, I get

$v = \frac{1}{2} y^2 + h[x]$ and $v_x = \frac{dh}{dx}$

A comparison with the last v_x with the expressions in the yellow cell shows that $\frac{dh}{dx} = -x$ or $h[x] = -\frac{1}{2} x^2$

Thus the following results:

```
f[z] = u + ⅈ v = x y + ⅈ (1/2 y^2 + -1/2 x^2 + C)
```

Set::write: Tag Plus in u+ⅈ v is Protected.≫

$$x\,y + i\left(C - \frac{x^2}{2} + \frac{y^2}{2}\right)$$

```
out = Simplify[x y + ⅈ (1/2 y^2 + -1/2 x^2 + C)]
```

$$i\,C - \frac{1}{2}\,i\,(x + i\,y)^2$$

```
out1 = out /. (x + ⅈ y) → z
```

$$i\,C - \frac{i\,z^2}{2}$$

```
Solve[-1/2 ⅈ (z^2 + c) == ⅈ C - (ⅈ z^2)/2, c]
```

$$\left\{\left\{c \to -\frac{C}{2}\right\}\right\}$$

The green cell above matches the text answer, modified by the value of C (real) shown in the purple cell.

15. $u = \frac{x}{x^2 + y^2}$

```
Clear["Global`*"]
```

```
u[x_, y_] = x/(x^2 + y^2)
```

$$\frac{x}{x^2 + y^2}$$

```
Simplify[Laplacian[u[x, y], {x, y}]]
```

0

The function u passes the test for harmonic function. Now to look for a corresponding analytic function.

```
D[u[x, y], x]
```

$$-\frac{2\,x^2}{\left(x^2 + y^2\right)^2} + \frac{1}{x^2 + y^2}$$

```
-D[u[x, y], y]
```

$$\frac{2\,x\,y}{\left(x^2+y^2\right)^2}$$

$$v_y = u_x = -\frac{2\,x^2}{\left(x^2+y^2\right)^2} + \frac{1}{x^2+y^2} \quad \text{and} \quad v_x = -u_y = \frac{2\,x\,y}{\left(x^2+y^2\right)^2}$$

Integrating the first equation with respect to y and differentiating the result with respect to x, I get

$$\text{vup} = \int\left(-\frac{2\,x^2}{\left(x^2+y^2\right)^2} + \frac{1}{x^2+y^2}\right) dy$$

$$-\frac{y}{x^2+y^2}$$

```
vup2 = vup + h[x] + C
```

$$C - \frac{y}{x^2+y^2} + h[x]$$

$$v_x = D[\text{vup2}, x]$$

$$\frac{2\,x\,y}{\left(x^2+y^2\right)^2} + h'[x]$$

A comparison with the expressions in the yellow cell shows that $\frac{dh}{dx} = 0$ or h[x] = C1.

Thus f[z] = u[x, y] + i v[x, y] and (making C2=C+C1)

$$f[z] = \frac{x}{x^2+y^2} + i\left(-\frac{y}{x^2+y^2} + C2\right)$$

$$\frac{x}{x^2+y^2} + i\left(C2 - \frac{y}{x^2+y^2}\right)$$

```
f1[z] = Simplify[f[z]]
```

$$\frac{1 + i\,C2\,x - C2\,y}{x + i\,y}$$

```
f2[z] = f1[z] /. (x + i y) → z
```

$$\frac{1 + i\,C2\,x - C2\,y}{z}$$

$$f3[z] = \frac{1 + i\,C2\,(x + i\,y)}{z} == \frac{1 + i\,C2\,z}{z} == \frac{1}{z} + i\,C2;$$

This answer does not match the text because a real constant C is left sitting next to an imaginary unit. Since the text answer is $\frac{1}{z}$+C, C real, I think it would be an abuse to show green here. Or maybe not. If C is is chosen as 0, then the answer can be green.

17. $v = (2x + 1)y$

This one has the twist of looking for u instead of the usual v.

```
Clear["Global`*"]

v[x_, y_] = (2 x + 1) y
(1 + 2 x) y

Simplify[Laplacian[v[x, y], {x, y}]]
0
```

The function v passes the test for harmonic function. Now to look for a corresponding analytic function.

```
-D[v[x, y], x]
-2 y

D[v[x, y], y]
1 + 2 x
```

$u_x = v_y = 1 + 2x$ and $u_y = -v_x = -2y$

Integrating the first equation with respect to x and differentiating the result with respect to y, I get

```
up = ∫(1 + 2 x) ⅆx
x + x^2

up2 = up + h[y] + c
c + x + x^2 + h[y]

u_y = D[up2, y]
h′[y]
```

A comparison with the expressions in the yellow cell shows that $\frac{dh}{dy} = -2y$ or $h[y] = -y^2$. Thus

f[z] = u[x, y] + i v[x, y] and

```
f[z] = x + x^2 + c - y^2 + ⅈ ((2 x + 1) y)
c + x + x^2 + ⅈ (1 + 2 x) y - y^2
```

```
f1[z] = FullSimplify[f[z]]
c + (x + ⅈ y) (1 + x + ⅈ y)
```

```
f2[z] = f1[z] /. (x + ⅈ y) → z
```

```
c + z (1 + z)
```

19. $v = e^{x}$ Sin[2 y]

Again my quarry is the u function instead of the v function.

```
Clear["Global`*"]
```

```
v[x_, y_] = ⅇ^x Sin[2 y]
ⅇ^x Sin[2 y]
```

```
Simplify[Laplacian[v[x, y], {x, y}]]
```

```
-3 ⅇ^x Sin[2 y]
```

The green cell above is not 0; therefore the function is not harmonic.

21 - 24 Determine a and b so that the given function is harmonic and find a harmonic conjugate.

21. $u = e^{\pi x}$ Cos[a v]

This looks pretty intimidating as written; I'm going to start by assuming a typo, and insert y for v.

```
Clear["Global`*"]
```

```
u[x_, y_] = ⅇ^(π x) Cos[π y]
ⅇ^(π x) Cos[π y]
```

```
Simplify[Laplacian[u[x, y], {x, y}]]
0
```

It looks like **a** needs to equal π in order to have a harmonic function.

```
D[u[x, y], x]
```

```
ⅇ^(π x) π Cos[π y]
```

```
-D[u[x, y], y]
```

```
ⅇ^(π x) π Sin[π y]
```

$v_y = u_x = e^{\pi x} \pi \,\text{Cos}[\pi y]$ and $v_x = -u_y = e^{\pi x} \pi \,\text{Sin}[\pi y]$

Integrating the first equation with respect to y and differentiating the result with respect to x, I get

```
vup = ∫(e^(π x) π Cos[π y]) dy
```

```
e^(π x) Sin[π y]
```

As usual Mathematica neglects to insert a constant of integration. However, in this case the omission lands on the text answer.

```
vup2 = vup + h[x]
h[x] + e^(π x) Sin[π y]
```

```
vx = D[vup2, x]
e^(π x) π Sin[π y] + h′[x]
```

A comparison of v_x with the expressions in the yellow cell shows that $\frac{dh}{dx} = 0$ or $h[x] = C$. Thus $f[z] = u[x, y] + i\, v[x, y]$ and

```
f[z] = e^(π x) Cos[π y] + i (e^(π x) Sin[π y] + c)
e^(π x) Cos[π y] + i (c + e^(π x) Sin[π y])
```

The green cell above agrees with the text answer for v[x,y]. However, for f[z], I believe a constant has to come in there. Unless I was wrong about the typo, and (due to principles not understood by me) that accounts for the text dispensing with the constant.

23. $u = a x^3 + b x y$

```
Clear["Global`*"]
```

```
u[x_, y_] = a x^3 + b x y
a x^3 + b x y
```

```
Simplify[Laplacian[u[x, y], {x, y}]]
6 a x
```

It appears that **a** must equal zero for the function to be harmonic.

```
u[x_, y_] = b x y
```

```
b x y
```

```
Simplify[Laplacian[u[x, y], {x, y}]]
0
```

```
D[u[x, y], x]
```

b y

```
-D[u[x, y], y]
```

-b x

$v_y = u_x = b\,y$ and $v_x = -u_y = -b\,x$

Integrating the first equation with respect to y and differentiating the result with respect to x, I get

```
vup = ∫b y ⅆy
```

$\frac{b\,y^2}{2}$

```
vup2 = vup + h[x] + c
```

$c + \frac{b\,y^2}{2} + h[x]$

```
vx = D[vup2, x]
```

h′[x]

A comparison with the expressions in the yellow cell shows that $\frac{dh}{dx}$ =.
$-b\,x$ or $h[x] = -\frac{b}{2}x^2$

Thus f[z] = u[x, y] + i v[x, y] and

```
f[z] = b x y + ⅈ[b y^2/2 - b x^2/2 + c]
```

$b\,x\,y + i\left[c - \frac{b\,x^2}{2} + \frac{b\,y^2}{2}\right]$

```
f1[z] = Simplify[f[z]]
```

$b\,x\,y + i\left[c + \frac{1}{2}b\left(-x^2 + y^2\right)\right]$

The green cell matches the text answer. I noticed this when I saw that only v is covered in the answer, not f[z].

25. CAS Project. Equipotential Lines. Write a program for graphing equipotential lines u=const of a harmonic function u and of its conjugate v on the same axes. Apply the program to (a) $u = x^2 - y^2,\ v = 2\,x\,y,$ (b) $u = x^3 - 3\,x^2\,y - y^3,\ v = 3\,x^2\,y - y^3.$

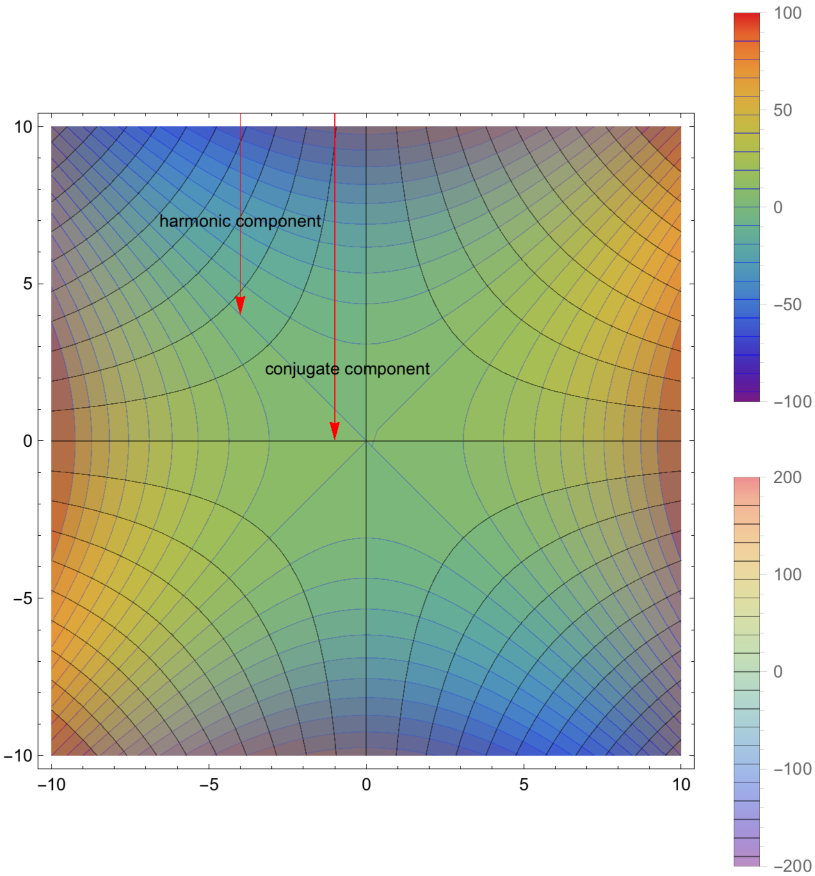
Part (a)

In a spooky coincidence, the exact same equations for part (a) were the subject of discussion in Mathematica StackExchange question #153214.

```
cp1 = ContourPlot[x^2 - y^2, {x, -10, 10},
   {y, -10, 10}, Contours → 20, PlotLegends → Automatic,
   ColorFunction → "Rainbow", ContourStyle → Blue,
   Epilog → {{Red, Arrowheads[.03], Arrow[{{-4, 11}, {-4, 4}}]},
     {Text["harmonic component", {-4, 7}]},
     {Red, Arrowheads[.03], Arrow[{{-1, 11}, {-1, 0}}]},
     {Text["conjugate component", {-0.6, 2.3}]}}];
```

```
cp2 = ContourPlot[2 x y, {x, -10, 10},
   {y, -10, 10}, Mesh → None, (*ContourShading→None,*)
   ColorFunction → Function[f, Opacity[.5, ColorData["Rainbow"][f]]],
   Contours → 20, PlotLegends → Automatic];
```

```
Show[cp1, cp2 /. EdgeForm[] → EdgeForm[Opacity[0]]]
```



Part (b)

```
Clear["Global`*"]
```

```
cp1 = ContourPlot[x^3 - 3 x^2 y - y^3, {x, -10, 10},
   {y, -10, 10}, Contours → 20, PlotLegends → Automatic,
   ColorFunction → "Rainbow", ContourStyle → Blue,
   Epilog → {{Red, Arrowheads[.03], Arrow[{{-4, 11}, {-4, 5.3}}]},
     {Text["harmonic component", {-4, 7}]},
     {Red, Arrowheads[.03], Arrow[{{0, 11}, {0, 5.75}}]},
     {Text["conjugate component", {0, 9}]}}];

cp2 = ContourPlot[3 x^2 y - y^3, {x, -10, 10},
   {y, -10, 10}, Mesh → None, (*ContourShading→None,*)
   ColorFunction → Function[f, Opacity[.5, ColorData["Rainbow"][f]]],
   Contours → 20, PlotLegends → Automatic];

Show[cp1, cp2 /. EdgeForm[] → EdgeForm[Opacity[0]]]
```